MAT 1341D Quiz 2 Solutions

11 February 2019

Family name: ____________________

First name: ____________________

Student number: ____________________

DGD section: ____________________

1	F
2	C
3	A
4	C
Total	

Please read these instructions very carefully:

1. Read each question carefully, and respond to each question **both on the question page and in the space provided above.**

2. You are not allowed to use calculators, phones, or any other electronic devices during the quiz; nor are you allowed to consult any notes or books.

3. Each of the four questions is multiple choice and worth 1 point. **You must justify your answer** on the question page, and record your answer both on that page and on the title page above. Answers without justification will not earn points.

1. Which of the following subsets of vectors form a basis of $\mathbb{R}^3$?

1. {(1,6,5), (1,4,1), (1,3,-1)}
2. {(-1,2,3), (3,3,2) }
3. {(-1,3,-5),(1,-2,4),(2,0,4),(5,1,9)}

A. All three are bases.
B. Only (1) is a basis.
C. Only 2 is a basis.
D. (1) and (2) are both bases, but (3) is not.
E. (2) and (3) are both bases, but (1) is not.
F. None of these three are bases.

The answer is F. We know from class that $\mathbb{R}^3$ *has dimension 3 (it has the usual basis* $\{(1,0,0),(0,1,0),(0,0,1)\}$*), so every basis of* $\mathbb{R}^3$ *contains 3 vectors. Hence option 2 is too small to be a basis, and option 3 is too big.*

It remains to check whether option 1 is a basis. We will check linear independence first: we look for a solution to the equation

$$a(1,6,5)+b(1,4,1)+c(1,3,-1)=(0,0,0).$$

This leads to the system of equations

$$a+b+c=0$$

$$6a+4b+3c=0$$

$$5a+b-c=0.$$

The first equation yields $c=-(a+b)$*. Plugging this into the second equation yields* $3a+b=0$*, so* $b=-3a$*. Finally, the last equation becomes* $5a-3a-2a=0 \Rightarrow 0=0$*. Hence there are infinitely many solutions (one for each value of* a*. Thus (1) is LD, and so cannot be a basis.*

2. Consider the set $W = \{(x, y, z) \in \mathbb{R}^3 | x + y = 0\}$. Which of the following statements is true?
A. W is a subspace of $\mathbb{R}^3$ and $\dim W = 3$..
B. W is a subspace of $\mathbb{R}^2$ and $\dim W = 1$..
C. W is a plane in $\mathbb{R}^3$ which passes through the origin and is parallel to the z-axis.
D. W is not a subspace of $\mathbb{R}^3$.
E. W is a line which passes through the origin.
F. W is a plane in $\mathbb{R}^3$ which passes through the origin and is parallel to the x-axis.

The answer is C. The equation defining W *as a subset of* $\mathbb{R}^3$ *tells us it is a plane through the origin; thus it is a subspace of dimension 2, which eliminates every answer except C and F. Now note that the normal vector to the plane is* $(1, 1, 0)$*, which is orthogonal to the* z*-axis (the vector (0,0,1)) but not to the* $x-$*axis (the vector (1,0,0)). Hence the plane is parallel to the* z*-axis.*

3. Consider

$$S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \right\} \subset M_2(\mathbb{R}).$$

Which of the following statements about S is true?

I. S is linearly dependent.

II. $\operatorname{span}(S) = \operatorname{span}\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$.

III. The dimension of S is equal to 3.

IV. $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in \operatorname{span}(S)$.

A. (I) and (II)
B. (I) and (III)
C. (II) and (IV)
D. (II) and (III)
E. (I), (III) and (IV)
F. (III) and (IV).

The answer is A. To check (I), set up the usual equation:

$$a \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + c \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We get a system of three equations from this:

$$a + c = 0$$

$$a - b - c = 0$$

$$b + 2c = 0$$

Hence $a = b + c$. Plugging into the first equation gives $b = -2c$. Substituting this in the last equation yields $0 = 0$, so we have infinitely many solutions (one for each choice of the parameter c). Thus the set S is linearly dependent.

This immediately eliminates option (III) (in fact, option (III) was absurd to begin with: S is a finite set of vectors, not a vector space). Since we know option (I) is true, the answer must be A.

To double check that (II) is true, we note that (II) claims that the last matrix in S is "redundant" – already contained in the span of the other two vectors. So, we should try to write this third matrix in terms of the other 2. You could solve another system of equations, but in fact, some inspection will lead you to the answer

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}.$$

Finally, we could check (IV) (again, really, we could have stopped back in the second paragraph). We try to solve

$$a\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} + b\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}$$

(note that because of (II), we don't have to include the third matrix here). The resulting system is

$$a = 1$$

$$a - b = 1$$

$$b = 1.$$

This system clearly has no solutions, so (IV) is false.

4. Let V be a vector space of dimension 15 and suppose W is a subspace of V with spanning set $\{v_1, \ldots, v_7\}$.

Which of the following statements is ALWAYS true?

I. $\dim W < 7$
II. $\dim W \leq 15$
III. $\dim W > 6$
IV. Every linearly independent subset of W contains at most 7 vectors.

A. (I) and (II)
B. (I) and (III)
C. (II) and (IV)
D. (II) and (III)
E. (I), (III) and (IV)
F. (III) and (IV).

The answer is C. Option (I) is almost correct, but it's possible that $\{v_1, \ldots, v_7\}$ *is a basis as well as a spanning set, in which case we would have* $\dim W = 7$. *On the other hand, it's possible that* $\{v_1, \ldots, v_7\}$ *is highly linearly dependent, and that* $\dim W$ *is very small (we could just have that every other vector in this set is a multiple of* v_1, *for instance, in which case* W *only has dimension 1). Hence option (III) can't always be true from the given information.*

Option (II) is always true, because we learned that if W *is a subspace of* V, *we have* $0 \leq \dim W \leq \dim V$; *in this case,* $\dim W$ *is bounded between* 0 *and* 15. *Finally, option (IV) is true because every linearly independent subset of a vector space is smaller than every spanning set.*